Analytical Solution for Drainage from a Uniformly Wetted Deep Soil Profile

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Abstract
An analytical solution of the Richards equation for drainage of water from an initially uniformly wet homogeneous soil profile is discussed. This solution builds on an approach originally used for infiltration and later adapted to drainage. The mathematics for this solution are scattered and incomplete. Here we provide a complete unified solution. The results are presented for some soils from the HYDRUS-1D standard soil types. Results show progression of the draining front down through the soil with time and can be used to estimate the time for the soil to drain to a particular water content. Comparison of the analytical solution with HYDRUS-1D numerical solution shows the solution works well at small drainage times but estimates more drainage in the upper soil profile than HYDRUS-1D at larger times.

Key Words
Soil physics, Richards equation, modelling

Introduction
The drainage of water from a homogeneous soil is described by the Richards['] equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z} - K(\theta)\right] \tag{1}$$

where z is depth below the soil surface, t is time, and θ is volumetric soil water content. The soil water diffusivity and unsaturated conductivity are $D(\theta)$ and $K(\theta)$ respectively, and are strongly dependent on soil water content. The $D(\theta)$ is assumed to be of the form (Broadbridge and White, 1988a):

$$D(\Theta) = K_s C(C-1)) / \left[\alpha \Delta\theta (C-\Theta)^2\right] \tag{2}$$

where θ_s and θ_r are the saturated and residual soil water contents respectively and $\Delta\theta = \theta_s - \theta_r$ is their difference, $\Theta = (\theta - \theta_r)/\Delta\theta$ is the scaled water content, K_s is the saturated soil hydraulic conductivity, α is an inverse capillary length parameter of Philip (1985), and $C > 1$ is a dimensionless parameter. The hydraulic conductivity is assumed to have the functional form

$$K(\Theta) = K_s (C-1)\Theta^2 / (C-\Theta). \tag{3}$$

Broadbridge and White's (1988a,b) model for soil water movement combines the functional form of the diffusivity from Fujita (1952) and Knight and Philip (1974) with the Burgers' equation form of the conductivity used by Clothier *et al*. (1981). Saunders *et al*. (1988) also independently came up with a similar solution. The Broadbridge and White analytical solution uses a transformation found by Fokas and Yortsos (1982), and is a combination of the Storm (1951) solution used by Knight and Philip (1974) in the absence of gravitational effects and the Burgers' equation solution given by Clothier et al. (1981). Warrick *et al*. (1990) adapted the Broadbridge and White solutions for soil water drainage, and Parkin *et al*. (1995) and Si and Kachanoski (2000) further explored their use. These drainage solutions use different boundary conditions to the Broadbridge and White solution and result in different functions in the solution. The mathematics of the analytical solution is scattered and incomplete in the published literature; this document presents a unified account of the theory.
The solution for infiltration into a deep profile given by Broadbridge and White (1988a) and adapted for drainage in a deep soil by Warrick *et al*. (1990) and Parkin *et al*. (1995) had zero water flux at the soil surface;the initial and boundary conditions were; $\theta(z,0) = \theta_0$, $K(\theta) - D(\theta)\frac{\partial \theta}{\partial z} = 0$ at $z = 0$

$$\frac{\partial \theta}{\partial z} = 0, z \to \infty$$

The corresponding initial value for the scaled water content is $\Theta_0 = (\theta_0 - \theta_r)/\Delta\theta$. The transformations used by Broadbridge and White (1988) introduce a dimensionless time parameter $\tau = 4C(C-1)\alpha K_s t / \Delta\theta$, a new dimensionless space-like parameter ς and a new independent variable $u(\varsigma,\tau)$ which satisfies a linear advection diffusion equation. It is important to note that the solutions $u(\varsigma,\tau)$ for the infiltration case studied by Broadbridge and White (1988) and the drainage case studied by Warrick et al. (1990) are different because of their different boundary conditions.

The Warrick *et al*. (1990) solution for $u(\varsigma,\tau)$ for drainage with zero surface flux is given by

$$u(\varsigma,\tau) = \exp\left(-\varsigma^2/\tau\right)\left[f\left(\varsigma/\sqrt{\tau}\right) + \tfrac{1}{2} f\left(\left[\tfrac{1}{2}B_0\tau - \varsigma\right]/\sqrt{\tau}\right) - \tfrac{1}{2} f\left(\left[\tfrac{1}{2}B_0\tau + \varsigma\right]/\sqrt{\tau}\right)\right] \quad (4)$$

with $f(x) \equiv \exp(x^2)\operatorname{erfc}(x)$, and the parameter B_0 given in terms of the initial condition by $B_0 = \Theta_0/(C-\Theta_0)$. For the case of no surface flux the original space variable is given in terms of the solution $u(\varsigma,\tau)$ by $z(\varsigma,\tau) = \{\varsigma - \ln[u(\varsigma,\tau)]\}/(C\alpha)$ and the scaled water content is given by $\Theta(\varsigma,\tau) = C\left[1 - \left(1 - \frac{\partial u}{\partial\varsigma}(\varsigma,\tau)\Big/u(\varsigma,\tau)\right)^{-1}\right]$. The quantity $\frac{\partial u}{\partial\varsigma}(\varsigma,\tau)\Big/u(\varsigma,\tau)$ does not seem to be given explicitly in the published literature, and is

$$\frac{\frac{\partial u}{\partial\varsigma}(\varsigma,\tau)}{u(\varsigma,\tau)} = -B_0 f\left[\left(\frac{\frac{1}{2}B_0\tau - \varsigma}{\sqrt{\tau}}\right) + f\left(\frac{\frac{1}{2}B_0\tau + \varsigma}{\sqrt{\tau}}\right)\right]\left[2f\left(\frac{\varsigma}{\sqrt{\tau}}\right) + f\left(\frac{\frac{1}{2}B_0\tau - \varsigma}{\sqrt{\tau}}\right) - f\left(\frac{\frac{1}{2}B_0\tau + \varsigma}{\sqrt{\tau}}\right)\right]^{-1}. \quad (5)$$

The surface value of the scaled water content is then

$$\Theta(0,\tau) = C\left[1 - \left(1 - \frac{\partial u}{\partial\varsigma}(0,\tau)\Big/u(0,\tau)\right)^{-1}\right] = C\left[1 - \left(1 + B_0 f\left(\tfrac{1}{2}B_0\sqrt{\tau}\right)\right)^{-1}\right]. \quad (6)$$

When $\tau = 0$, then eqn (6) becomes $\Theta(0,0) = C\left[1 - (1 + B_0)^{-1}\right] = \Theta_0$ as required.

Equations 4-6 provide a unified solution which is easily computed. We used the eqs 4-6 above to model drainage from a uniformly wet soil profile using soil properties taken from HYDRUS-1D soil catalogue and are presented in Table 1. The desorptivity was estimated from simulation of desorption of an initially saturated horizontal soil columns using HYDRUS-1D (Simunek *et al*. 1998) and C and α by fitting the water content, potential relationship to eqn (25) of Broadbridge and White (1988b). We also calculated results for the Brindabella silty clay loam (not shown) to check that we got the same results as Warrick *et al*. (1990). Eqs 4-6 were solved using MatLab.

Table 1. Soil properties derived from HYDRUS-1D catalogue.

Soil	θ_r (m^3 m^{-3})	θ_s (m^3 m^{-3})	C	K_s (m s^{-1})	α (m^{-1})
Clay	0.068	0.38	1.0002	5.56x10^{-7}	6.92
Silt	0.078	0.46	1.0063	6.94x10^{-7}	5.15
Loam	0.078	0.43	1.0189	2.89x10^{-6}	7.11
Sand	0.045	0.43	1.0458	8.35x10^{-5}	17.94

Results and Discussion

The drainage of the soil from saturation with a free drainage (unit gradient) boundary condition was simulated in HYDRUS-1D for the soils in Table 1 and compare with drainage calculated with the analytical solution (Fig.1). The analytical solution appears to estimate more drainage near the surface especially at large times than does HYDRUS-1D. For times less than 10 days the analytical solution and HYDRUS-1D give similar ($r^2 > 0.88$) estimates of the water content profile. The percentage difference in the water content at the surface between the analytical solution and HYDRUS-1D increase with time from negligible to approximately 28% by $t = 50$ days, except for the sand. This is most likely due to different functional relationships for K and D used in the analytical solution and HYDRUS-1D. This will be explored in the future.

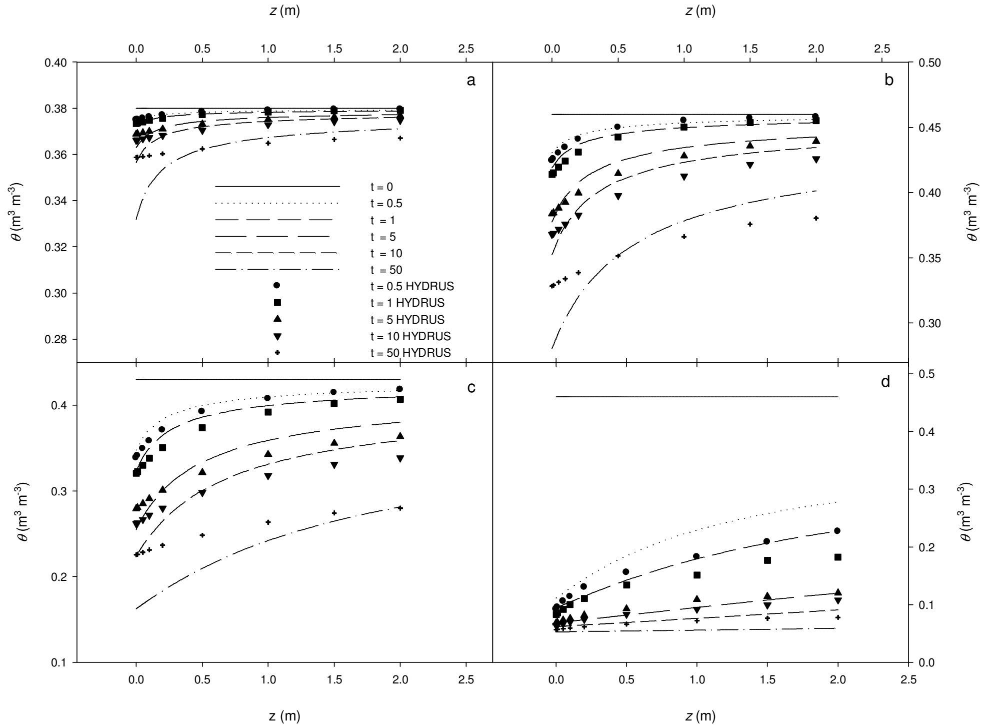

Figure 1. Comparison of the analytical solution with HYDRUS-1D for drainage of an initially saturated soil profile for a) clay, b) silt, c) loam and d) sand. The times indicated in the legend are days since drainage commenced.

The drainage from different initial uniform water content profiles was calculated with the analytical solution and HYDRUS-1D (Fig. 2). This indicates that the percentage difference between the analytical solution and the numerical solution at large times (t = 50 days) is approximately the same, as the initial water content decreases.

Conclusion

An analytical solution for drainage from uniformly wet soil profiles is shown to give similar water content profiles ($r^2 > 0.88$) at drainage times < 10 days to those simulated with a one-dimensional numerical solution of Richards' equation. This is especially so at the surface where the percentage difference in the water content between the two models for all but the sand soils increases with increasing time. At large times the analytical solutions water content near the surface is usually less than the numerical solution. The analytical solution may overestimate drainage at large times.

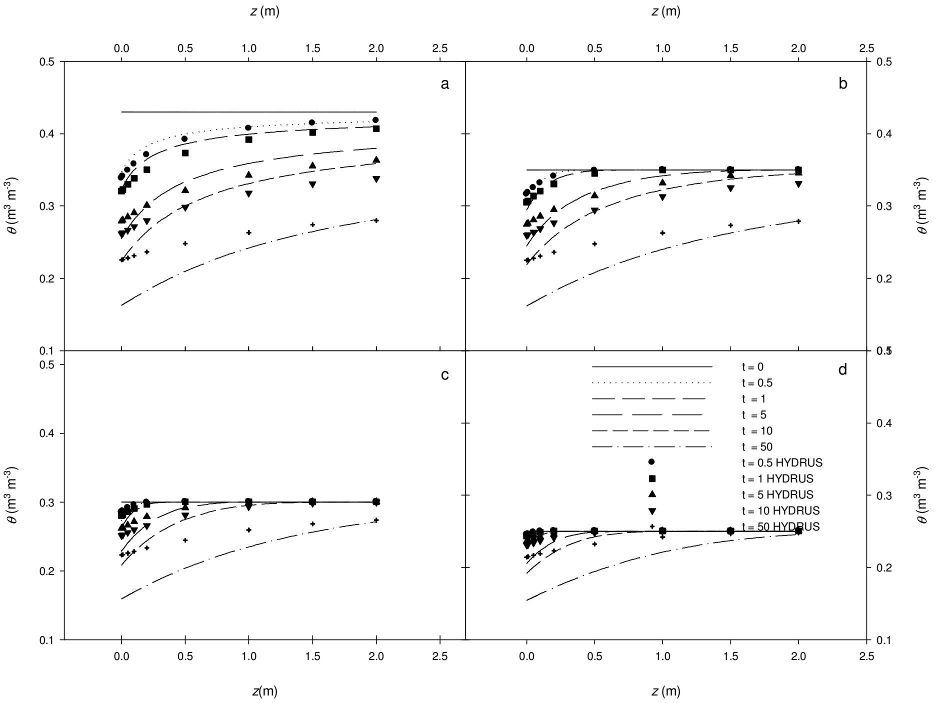

Figure 2. Comparison of the analytical solution with HYDRUS-1D for drainage of an initially saturated loam soil profile for a) $\theta = 0.43$, b) $\theta = 0.35$, c) $\theta = 0.30$ and d) $\theta = 0.25$. The times indicated in the legend are days since drainage commenced.

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